International Journal of Theoretical Physics, Vol. 40, No. 10, October 2001 (© 2001)

Proper Effect Algebras Admitting No States

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We show that there is even a finite proper effect algebra admitting no states. Further, every lattice effect algebra with an ordering set of valuations is an MV effect algebra (consequently it can be organized into an MV algebra). An example of a regular effect algebra admitting no ordering set of states is given. We prove that an Archimedean atomic lattice effect algebra is an MV effect algebra iff it admits an ordering set of valuations. Finally we show that every nonmodular complete effect algebra with trivial center admits no order-continuous valuations.

1. INTRODUCTION AND BASIC DEFINITIONS

Modular orthocomplemented lattices (Birkhoff and Von Neumann, 1936) and orthomodular lattices (Kalmbach, 1983) are measure-carrying structures that arise in physical theories as the carrier of quantum mechanical probabilities (see also Pták and Pulmannová, 1991; Riečan and Neubrunn, 1997). Effect algebras are partial algebras (originally of positive self-adjoint operators lying between the zero operator and the identity operator on Hilbert space) that arise in the theory of quantum measurements as the structure in which a classical law of noncontradiction ($p \wedge p' = 0$) could fail, thus allowing for the possibility of unsharp or fuzzy propositions (Bennett and Foulis, 1994; Kôpka, 1992; Kôpka and Chovanec, 1994).

Many-valued logics (MV algebras) is a way of introducing more than two values into modal logics (Chang, 1958).

Lattice effect algebras (or D lattices) give a common generalization of orthomodular lattices (including Boolean algebras) and MV algebras.

Definition 1.1. (Foulis and Bennett, 1994). A partial algebra $(E; \oplus, 0, 1)$ is called an *effect algebra* if 0, 1 are two distinguished elements and $\oplus$ is a partially defined binary operation on E that satisfies the following conditions for

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0020-7748/01/1000-1683$19.50/0

any $a, b, c \in E$:

(Ei) $b \oplus a = a \oplus b$ if $a \oplus b$ is defined,
(Eii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if one side is defined,
(Eiii) for every $a \in P$ there exists a unique $b \in P$ such that $a \oplus b = 1$ (we put $a' = b$),
(Eiv) if $1 \oplus a$ is defined, then $a = 0$.

We often denote the effect algebra $(E; \oplus, 0, 1)$ briefly by E. In every effect algebra E we can define the partial operation $\ominus$ and the partial ordder $\leq$ by putting

$$a \leq b \quad \text{and} \quad b \ominus a = c \quad \text{iff} \quad a \oplus c \text{ is defined} \quad \text{and} \quad a \oplus c = b.$$

Since $a \oplus c = a \oplus d$ implies $c = d$, the $\ominus$ and the $\leq$ are well defined. If E with the defined partial order is a lattice (a complete lattice) then $(E; \oplus, 0, 1)$ is called a *lattice effect algebra* (a *complete effect algebra*).

A subset Q with inherited operation $\oplus$ is called a *subeffect algebra* of E iff (i) $1 \in Q$, (ii) if out of elements $a, b, c \in E$ with $a \oplus b = c$ at least two are in Q then $a, b, c \in Q$.

It is worth noting that if $(E; \oplus, 0, 1)$ is an effect algebra then $(E; \ominus, 0, 1)$ with the partial binary operation $\ominus$ defined above is a *D poset* introduced by Kôpka and Chovanec (1994) and vice versa (see also Dvurečenskij and Pulmannová, 2000; Riečan and Neubrunn, 1997).

Definition 1.2. Assume that $(E; \oplus, 0, 1)$ is an effect algebra. A map $m : E \to [0, 1]$ is called a (finitely additive) *state* on E if $m(1) = 1$ and $a \leq b' \Rightarrow m(a \oplus b) = m(a) + m(b)$. We say that m is *faithful* if $m(a) = 0 \Rightarrow a = 0$.

Definition 1.3. A state m on a lattice effect algebra $(E; \oplus, 0, 1)$ is called a *valuation* if for $a, b \in E$, $a \wedge b = 0 \Rightarrow m(a \vee b) = m(a) + m(b)$.

Note that if m is a state on an effect algebra E then for $a, b \in E$ with $a \leq b$ we have $b = a \oplus (b \ominus a)$, which implies $m(b) = m(a) + m(b \ominus a)$. Thus $a \leq b \Rightarrow m(a) \leq m(b)$ and $m(b \ominus a) = m(b) - m(a)$.

If ω is a valuation on a lattice effect algebra E then evidently $\omega(a \vee b) \leq \omega(a) + \omega(b)$ for all $a, b \in E$ (we say that ω is *subadditive*). On the other hand a state on a lattice effect algebra need not be subadditive. In Riečanová (in press a) it has been shown that a state ω on a lattice effect algebra E is valuation iff ω is subadditive. Moreover, if ω is a valuation then for all $a, b \in E$ we have $\omega(a \vee b) + \omega(a \wedge b) = \omega(a) + \omega(b)$.

The aim of this paper is to bring an example of a proper effect algebra admitting no states and a proper regular effect algebra admitting no ordering set of states. Moreover, we establish a relation between MV effect algebras (MV algebras) and the existence of ordering sets of valuations.

For basic properties of effect algebras and MV effect algebras we refer to Dvurečenskij and Pulmanová (2000).

2. EXAMPLE OF A PROPER EFFECT ALGEBRA ADMITTING NO STATES

R. Greechie (1971) has shown that there are even finite orthomodular lattices admitting no states. On the other hand for every separable complete modular atomic ortholattice there exists a faithful order-continuous state, actually a probability measure being a valuation (Riečanová, 1998; Kirchheimová and Riečanová, 1997). The last result has been extended by Riečanová (in press a) to complete modular atomic effect algebras.

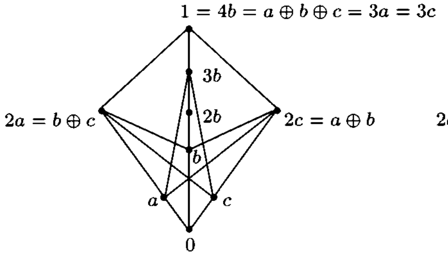
Recall that an orthomodular lattice $(L, \vee, \wedge, ', 0, 1)$ becomes a lattice effect algebra if for $a, b \in E$ we say that $a \oplus b$ is defined iff $a \leq b'$, and then we put $a \oplus b = a \vee b$. Thus Greechie's example (Greechie, 1971) provide an example of a lattice effect algebra admitting no states.

Effect algebras with respect to $\oplus$ operation and lattice operations $\vee$ and $\wedge$ possessed some asymmetry. Namely, for elements a, b of effect algebra E

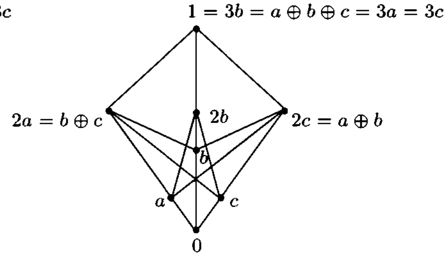
(1) if $a \oplus b$ and $a \vee b$ exists in E then $a \wedge b$ also exists in E,
(2) the existence of $a \oplus b$ and $a \wedge b$ in E does not imply the existence of $a \vee b$.

Definition 2.1. (Riečanová, 1997). An effect algebra $(E; \oplus, 0, 1)$ is called proper if there is a pair $a, b \in E$ such that $a \oplus b$ and $a \wedge b$ exist in E but $a \vee b$ does not exist in E.

Riečanová (in press b) has shown that an effect algebra E is proper iff there is a pair $a, b \in E$ such that $a \wedge b = 0$, and $a \oplus b$ is defined in E but $a \vee b$ does not exist. S. Gudder (2000, personal communication) proved that all real and complex Hilbert space effect algebras of dimensions greater than one are proper.



Example 2.3



Example 3.2

It is natural to ask whether proper effect algebras can be embedded into lattice effect algebras. Riečanová (1997) showed that *a proper effect algebra cannot be a dense subeffect algebra of a lattice effect algebra*. Here, we say that an effect algebra E_1 is a *dense subeffect algebra* of an effect algebra E_2 iff to every nonzero element $x \in E_2$ there is a nonzero element $y \in E_1$ with $y \leq x$. In this case all suprema and infima of subsets of E_1 existing in E_2 are preserving for E_1 (Riečanová, 1997).

Next example is an extension of Greechie's result mentioned above to proper effect algebras. For proper orthoalgebras we refer to Hamhalter, Navara and Pták, 1995.

Proposition 2.2. *There is a finite proper effect algebra admitting no states.*

Example 2.3. Let us consider the effect algebra $(E; \oplus, 0, 1)$ with $E = \{0, a, b, c, 2a, 2b, 2c, 3b, 1\}$ and $1 = a \oplus b \oplus c = 3a = 4b = 3c$. Evidently, $a \vee b$ does not exist in E but $a \wedge b = 0$ and $a \oplus b = 2c$. It follows that E is a proper effect algebra.

Assume that $m\colon E \to \langle 0, 1 \rangle$ is a state on E. The equality $1 = 4b = 3a = 3c$ together with the condition $m(1) = 1$ imply that $m(a) = m(c) = \frac{1}{3}$ and $m(b) = \frac{1}{4}$. Moreover, the equality $1 = a \oplus b \oplus c$ implies that $m(a) + m(b) + m(c) = 1$, a contradiction.

3. REGULAR EFFECT ALGEBRAS ADMITTING NO ORDERING SET OF STATES

Definition 3.1. A set $\mathcal{M}$ of states on an effect algebra $(E; \oplus, 0, 1)$ is called *order determining* (or *ordering*, for brevity) iff for all $a, b \in E$ the condition $m(a) \leq m(b)$ for all $m \in \mathcal{M}$ implies that $a \leq b$.

Recall, that an element a of an effect algebra E is called *isotropic* iff $2a = a \oplus a$ is defined in E and E is called *regular* iff any two isotropic elements $c, d \in E$ are *orthogonal* (equivalently $c \oplus d$ is defined). Any Boolean algebra carries an ordering set of states, as does the standard scale effect algebra $[0, 1] \subseteq \mathbb{R}$ (the real unit interval in which $p \oplus q = p + q$ iff $p + q \leq 1$) and the standard effect algebra $\mathcal{E}(H)$ of all positive self-adjoint operators on a Hilbert space H that are bounded above by identity operator. Every *effect algebra admitting an ordering set of states* is regular. Thus standard effect algebra $[0, 1] \subseteq \mathbb{R}$ and $\mathcal{E}(H)$ are regular effect algebras (Foulis, n.d.).

We bring an example of even finite regular effect algebra admitting no ordering set of states.

Example 3.2. Let $(E; \oplus, 0, 1)$ be the effect algebra with $E = \{0, 1, a, b, c, 2a, 2b, 2c, 1\}$, in which $1 = 3a = 3b = 3c = a \oplus b \oplus c$. Then by cancellation law (see Dvurečenskij and Pulmannová, 2000), we have $b \oplus c = 2a$, $a \oplus b = 2c$, and $a \oplus c = 2b$.

Assume that $m: E \to \langle 0, 1\rangle$ is a state on E. Then $m(a) = m(b) = m(c) = \frac{1}{3}$, which gives that E admits no ordering set of states.

Note that E is proper and admits the unique state m: $m(a) = m(b) = m(c) = \frac{1}{3}$ and $m(2a) = m(2b) = m(2c) = \frac{2}{3}$. Of course, $m(0) = 0$ and $m(1) = 1$.

Proposition 3.3. *Let* $(E; \oplus, 0, 1)$ *be an effect algebra and let there be mutually different elements* $a, b, c \in E$ *such that* $a \oplus b = 2c$ *and* $b \oplus c = 2a$. *Then* E *admits no ordering set of states.*

Proof: Assume that $m: E \to \langle 0, 1\rangle$ is a state on E. Then $m(a) + m(b) = 2m(c)$ and $m(b) + m(c) = 2m(a)$. It follows that $m(a) = m(c)$. Thus the assumption that there is an ordering set $\mathcal{M}$ of states on E implies that $a = c$, a contradiction. □

4. ORDERING SET OF VALUATIONS

Assume that $(E; \oplus, 0, 1)$ is a lattice effect algebra. According to Chovanec and Kôpka (1997), elements a, b of a lattice effect algebra $(E; \oplus, 0, 1)$ are called *compatible* (written $a \leftrightarrow b$) iff $(a \vee b) \ominus a = b \ominus (a \wedge b)$. We say that $M \subseteq E$ is a *set of mutually compatible elements* iff any two elements of M are compatible. A lattice effect algebra is called an *MV effect algebra* (or a *Boolean effect algebra*) if any two elements of E are compatible.

The following statements may be found in Chovanec and Kôpka (1997), Riečanová (in press a), and Bennett and Foulis (1995).

Lemma 4.1. *Let* $(E; \oplus, 0, 1)$ *be a lattice effect algebra. Then the following conditions are equivalent:*

(i) $a \wedge b = 0 \Rightarrow a \le b'$.
(ii) *Any two elements a, b in E are compatible.*
(iii) $\oplus$ *has the unique extension to a semigroup operation* $\hat{\oplus}$ *on E s.t.* $(E; \hat{\oplus}, ', 0, 1)$ *becomes an MV algebra.*

An example of a lattice effect algebra that is neither an orthomodular lattice nor an MV algebra is, for example, a direct product or 0–1 pasting (pasting by identification of elements 0 and 1) of an orthomodular lattice and MV effect algebra that are considered as two effect algebras.

Theorem 4.2. *If a lattice effect algebra* $(E; \oplus, 0, 1)$ *admits an ordering set S of valuations then E is an MV effect algebra.*

Proof: Let $a, b \in E$ with $a \wedge b = 0$. Then $a' \vee b' = 1$, and hence for every $\omega \in S$ we have

$$1 + \omega(a' \wedge b') = \omega(a' \vee b') + \omega(a' \wedge b') = \omega(a') + \omega(b'),$$

which gives that

$$0 \leq \omega(a' \wedge b') = \omega(b') - (1 - \omega(a')) = \omega(b') - \omega(a).$$

It follows that $\omega(a) \leq \omega(b')$ for all $\omega \in S$ and hence $a \leq b'$. This proves that E is an MV effect algebra. □

Recall that an effect algebra is called *Archimedean* if for no nonzero element $e \in E$, $ne = a \oplus \cdots \oplus e$ (n times) exists for all $n \in N$.

Theorem 4.3. *An Archimedean atomic lattice effect algebra $(E; \oplus, 0, 1)$ is an MV effect algebra iff there is an ordering set S of valuations on E.*

Proof: (1) Let E be an atomic MV effect algebra and $A = \{a \in E \mid a$ is an atom of $E\}$. Define, for every $a \in A$, a map $\omega_a \colon E \to [0, 1]$ by $\omega_a(x) = \frac{k_a^x}{n_a}$, $x \in E$; where $n_a = \mathrm{ord}(a)$ is the greatest integer such that $n_a a = a \oplus \cdots \oplus a$ (n_a times) exists and $k_a^x \in N \cup \{0\}$ is the greatest integer for which $k_a^x a \leq x$. Evidently, $\omega_a(x) \leq 1$ for all $x \in E$ and $\omega_a(1) = 1$.

Assume that $x, y \in E$ with $x \leq y'$. We can easily see that the Riesz decomposition property implies that if $ka \leq x \oplus y$ then $ka = (la) \oplus (ta)$, where $la \leq x$ and $ta \leq y$, $l, t \in N \cup \{0\}$. If follows that $\omega_a(x \oplus y) = \omega_a(x) + \omega_a(y)$, and hence ω_a is a state on E. Since E is an MV effect algebra ω_a is a valuation.

Let us show that $S = \{\omega_a \mid a \in A\}$ is an ordering set of valuations on E. Clearly, $x, y \in E$ with $x \leq y$ implies $\omega_a(x) \leq \omega_a(y)$ for all $\omega_a \in S$. Conversely, assume that $\omega_a(x) \leq \omega_a(y)$ for all $\omega_a \in S$. By Riečanová (2001a) for every $x \in E$ we have $x = \bigvee\{u \in E \mid u \leq x, u$ is a finite element of $E\}$, where $u \in E$ is called finite if there are not the different necessary atoms $a_1, \ldots, a_n \in E$ such that $u = a_1 \oplus \cdots \oplus a_n$. Moreover if for mutually different atoms $a, b \in E$ and $k, l \in N$ there is $(ka) \oplus (lb)$, then $(ka) \wedge (lb) = 0$ and $(ka) \vee (lb) = (ka) \oplus (lb)$. It follows that $x = \bigvee\{ka \mid ka \leq x, k \in N, a$ is an atom of $E\}$. If $\omega_a(x) \leq \omega_a(y)$ for every atom a of E then $ka \leq x$ implies that

$$\frac{k}{n_a} \leq \frac{k_a^x}{n_a} = \omega_a(x) \leq \omega_a(y) = \frac{k_a^y}{n_a}$$

which gives that $ka \leq y$. We conclude that $x \leq y$.

(2) If there is an ordering set of valuations on E then E is an MV effect algebra by Theorem 4.2. □

5. COMPLETE EFFECT ALGEBRAS ADMITTING NO ORDER-CONTINUOUS VALUATIONS

Assume that $\mathcal{E}$ is a directed set and $(x_\alpha)_{\alpha\in\mathcal{E}}$ is a net of elements of a lattice effect algebra $(E; \oplus, 0, 1)$. We write $x_\alpha \uparrow x$ if $x_{\alpha_1} \le x_{\alpha_2}$, for all $\alpha_1 \le \alpha_2$, and $x = \bigvee\{x_\alpha \mid \alpha \in \mathcal{E}\}$. A lattice effect algebra E is called *order continuous* if $x_\alpha \uparrow x$ implies $x_\alpha \wedge y \uparrow x \wedge y$, for all $x_\alpha, x, y \in E$. A state ω on E is called order continuous ((o)-continuous for brevity) if $x_\alpha \uparrow x$ implies $\omega(x_\alpha) \uparrow \omega(x)$, for all $x_\alpha, x \in E$. For more we refer to Kirchheimová and Riečanová (1997).

In Riečanová (in press a) it has been shown that if on a lattice effect algebra E there exists a faithful (o)-continuous valuation then E is separable, modular, (Grätzer, 1998), and (o)-continuous. In this section we study complete effect algebras with not necessarily faithful but (o)-continuous valuation.

Recall that an element z of a lattice effect algebra E is *central* if $x = (x \wedge z) \vee (x \wedge z')$ for all $x \in E$. The *center* $C(E)$ of E is the set of all central elements of E (see Greechie *et al.*, 1995; Riečanová, 2000).

Theorem 5.1. *Let* $(E; \oplus, 0, 1)$ *be a complete effect algebra and let* $\omega : E \to [0, 1]$ *be an (o)-continuous valuation. Then* $a_0 = \bigvee\{x \in E \mid \omega(x) = 0\}$ *is a central element of* E *and* $\omega(a_0) = 0$. *Moreover,* E *is isomorphic to the direct product* $[0, a_0] \times [0, a_0']$ *of effect algebras* $[0, a_0]$ *and* $[0, a_0']$ *under which* $\omega/[0, a_0']$ *is a faithful valuation on the complete modular order-continuous and separable effect algebra* $[0, a_0']$.

Proof: Let us put $B = \{x \in E \mid x \leftrightarrow y \text{ for all } y \in E\}$ and $E_S = \{v \in E \mid v \wedge v' = 0\}$. By Riečanová (2001a) $C(E) = E_S \cap B$.

Let us first prove that $a_0 \in B$. Assume $x \in E$. Because ω is a valuation we have $\omega(x \vee a_0) + \omega(x \wedge a_0) = \omega(x) + \omega(a_0)$. Set $E_0 = \{x \in E \mid \omega(x) = 0\}$. For every finite $F \subseteq E_0$ we put $x_F = \bigvee F$. Let $\mathcal{E} = \{F \subseteq E_0 \mid F \text{ is finite}\}$. Clearly, $\mathcal{E}$ is directed by set inclusion and $x_F \uparrow a_0$. As ω is (o)-continuous we obtain that $\omega(x_F) \uparrow \omega(a_0)$. Further if $F = \{x_1, x_2, \ldots, x_n\}$ then $\omega(x_F) \le \omega(x_1) + \omega(x_2) + \cdots + \omega(x_n) = 0$, by subadditivity of ω. We conclude that $\omega(a_0) = 0$. It follows that $\omega(x \wedge a_0) = 0$ for every $x \in E$ and hence $\omega(x \vee a_0) = \omega(x)$. The last equality implies that $\omega((x \vee a_0) \ominus x) = \omega(x' \ominus (x' \wedge a_0')) = 0$. Thus $x' \ominus (x' \wedge a_0') \le a_0$, which gives that $x' \ominus (x' \wedge a_0') \leftrightarrow a_0$. As also $x' \wedge a_0' \leftrightarrow a_0$, we conclude that $x' = (x' \wedge a_0') \oplus (x' \ominus (x' \wedge a_0')) \leftrightarrow a_0$ (see Riečanová, 2000) and hence $x \leftrightarrow a_0$. This proves that $a_0 \in B$.

Let us show now that $a_0 \in E_S$. Assume that $e \le a_0 \wedge a_0'$ for some $e \in E$. Then $e \oplus a_0$ is defined and $\omega(e) = 0$. As ω is a state we obtain $\omega(e \oplus a_0) = 0$, which gives $e \oplus a_0 \le a_0$, and hence $e = 0$. This proves that $a \wedge a_0' = 0$, and hence $a_0 \in E_S$ (Riečanová, 2001b).

Finally for nonzero $x \in E$ with $x \leq a_0'$ we have $\omega(x) \neq 0$, since $\omega(x) = 0$ implies that $0 \neq x \leq a_0 \wedge a_0'$, a contradiction. By Riečanová (in press a) we obtain that $[0, a_0']$ is a complete modular order-continuous and separable effect algebra. □

Corollary 5.2. *Let* $(E; \oplus, 0, 1)$ *be a nonmodular complete effect algebra with the center* $C(E) = \{0, 1\}$. *Then* E *is admitting no order-continuous valuations.*

ACKNOWLEDGMENT

This research was supported by Grant 1/7625/20 of MŠ SR.

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